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Critical exponents of self-avoiding walks on a family of truncated n -simplex lattices

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Received 3 November 1989, in final form 21 February 1990

Abstract. We have studied self-avoiding walks (SAWs) on a family of truncated n -simplex lattices which provide a family of fractals in which the fractal dimension \bar{d} can be varied to a wide range while the spectral dimension \tilde{d} is held almost fixed. By means of exact renormalisation group transformations, we have calculated the critical exponents ν , α and γ and the connectivity constant μ of the SAWs for $n = 5$ and 6. We propose an approximate theory for calculating the critical exponents of the SAWs on fractals which is expected to be accurate at large values of n . We show that the theory gives results which are in good agreement with exact values even for small values of n and leads to simple closed relations for the exponents. In particular, for the family of fractals considered here, $\alpha + \gamma = 2$. The results of the theory are also compared with various phenomenological proposals.

1. Introduction

In order to study the effect of excluded volume on equilibrium statistics of a polymer chain in a good solvent, a model of self-avoiding walks (SAWs) on a lattice was proposed [1]. Other lattice models such as random walks (RWs), self-attracting self-avoiding walks (SASAWs), true self-avoiding walks (TSAWs), trails and their silhouettes etc, [2, 3] were proposed from time to time to represent polymers in different regimes. These lattice models have been the focus of much attention in recent years because from a statistical mechanics viewpoint they serve as generic examples of analysing scaling and fractal properties [4, 5]. In this paper we concentrate on the critical behaviour of SAWs on fractals.

The critical properties of SAWs are known exactly in one and two dimensions [1, 6], at its upper critical dimensionality [4] and known to a high degree of accuracy in three dimensions [7-9]. In another development, Flory [10, 11] used mean-field arguments to reach the following formula

$$\nu = \frac{3}{d+2} \quad (d \leq 4) \quad (1.1)$$

where d is dimensionality of space and

$$\nu = \lim_{N \rightarrow \infty} \frac{1}{2} \frac{\ln \langle R^2 \rangle}{\ln N}. \quad (1.2)$$

Here $\langle R^2 \rangle$ is the mean square end-to-end distance, N the number of monomers and ν the end-to-end distance exponent (correlation length exponent for a magnetic system).

The Flory formula estimates accurately the value of ν for periodic (Euclidean) lattices. The intriguing simplicity and remarkable accuracy of (1.1) has stimulated considerable work on understanding the basis of the Flory theory. However, an exact equation for ν in the quantity $(4-d)$ already disagrees with (1.1) at first order [12]. An expression in $(d-1)$ which should also be exact, also disagrees with (1.1) at first order [13]. In general SAWs on disordered lattices present subtle difficulties which are not yet fully understood.

Fractals, [14] which may be considered intermediate between regular and disordered lattices, offer a class of systems where the consequences of the loss of translational invariance may be studied in detail. Some exact results have been found in recent years for the critical exponents of SAWs on fractals [15–17]. From these results one finds that, in contrast to regular lattices, the critical behaviour of SAWs on a fractal is influenced by a number of features of the fractal such as its Hausdorff's dimension \bar{d} , the spectral dimension \tilde{d} , the lacunarity, etc.

In this paper, we study SAWs on a family of truncated n -simplex lattices introduced by Dhar [18] and Nelson and Fisher [19]. The lattice is defined recursively. The graph of the zeroth-order truncated n -simplex lattice is a complete graph on $(n+1)$ points. The graph of the $(r+1)$ th-order lattice is obtained by replacing each of the vertices by the r th-order graph by a complete graph on n points. Each of the new n points is connected to one of the lines leading to the original vertex. The r is allowed to tend to infinity. The r th-order lattice has $(n+1)n^r$ vertices and $(n+1)n^{r+1}/2$ bonds. Each lattice point has coordination number n . The fractal and spectral dimensions of these lattices are given by [18]

$$\bar{d} = \frac{\ln n}{\ln 2} \quad (1.3)$$

and

$$\tilde{d} = \frac{2 \ln n}{\ln(n+2)} \quad (1.4)$$

respectively.

The case $n=1$ corresponds to an uninteresting case of mutually disconnected pairs of points, $n=2$ corresponds to a linear chain. For $n=3$ we get the truncated tetrahedron lattice [19]. For higher values of n the lattices are non-planar. As n is varied from 3 to ∞ , \bar{d} changes from 1.5850 to ∞ , but \tilde{d} from 1.3652 to 2. Thus the truncated n -simplex lattices provide a family of fractals in which \bar{d} can be varied to a wide range while keeping \tilde{d} almost 'fixed'. This is in contrast to a family of Sierpinski-type fractals [20] studied by Eležović *et al* [17] in which \bar{d} and \tilde{d} vary from 1.5850 to 2 and from 1.3652 to 2, respectively, as a characterising integer parameter b runs from 2 to ∞ [21, 22].

The paper is organised as follows. In section 2 we briefly describe, for the sake of completeness and fixing the notation, the standard statistical model of linear polymers and model them by connected graphs of N links embedded in a lattice of N_0 points, each link representing a monomer unit. In the critical phenomena language the infinite SAW is a critical $O(s)$ model with $s=0$ components [1]. Section 3 is devoted to calculation of the critical exponents for a truncated 5-simplex lattice using real space renormalisation group (RG) theory. The 6-simplex lattice is considered in section 4. We develop in section 5 an approximate theory for the critical exponents of the SAWs and compare the results with exactly known values wherever possible. The paper ends with a brief discussion given in section 6.

2. Lattice model of a linear polymer and critical exponents

It has been assumed (see, for example, McKenzie [23]) that the total number C_N of a distinct SAW of N steps and the total number P_N of distinct SAW loops of N steps embedded in a lattice of N_0 points and averaged over all possible positions of the starting point follow the following power law for very large N :

$$C_N \sim \mu^N N^{\gamma-1} \tag{2.1}$$

$$P_N \sim \mu^N N^{\alpha-3}. \tag{2.2}$$

Here μ is the connectivity constant and α and γ the associated critical exponents. The corresponding generating functions are defined to be

$$C(x) = \sum_{N=1}^{\infty} C_N x^N \tag{2.3}$$

$$P(x) = \sum_{N=1}^{\infty} P_N x^N \tag{2.4}$$

where x is a weight factor (fugacity) associated with each step of the walk. The leading singular terms of the above equations, when x approaches $1/\mu$ from below, are of the form

$$C(x) \sim (1-x)^{-\gamma} \tag{2.5}$$

$$P(x) \sim (1-x)^{2-\alpha}. \tag{2.6}$$

The above singular behaviour resembles the behaviour of the initial susceptibility and free energy of a magnetic system being close to its critical point. This remarkable resemblance between the SAW model and the s -component spin model in the limit $s \rightarrow 0$ was first noted by de Gennes [12].

The mean squared end-to-end distance $\langle R^2 \rangle$ for N steps is also expected to obey the power law

$$\langle R^2 \rangle \sim N^{2\nu}. \tag{2.7}$$

The corresponding generating function

$$R(x) = \sum_{N=2}^{\infty} \langle R_N^2 \rangle C_N x^N / C(x) \tag{2.8}$$

has the leading singular term

$$R(x) \sim (1-x)^{-2\nu} \tag{2.9}$$

which parallels the correlation length criticality of the model magnetic system.

In order to determine the critical exponents for the SAW, we adopt the real space renormalisation group (RG) approach [15]. The 3- and 4-simplex lattices have already been studied by Dhar [15] and Dhar and Vannimenus [24]. These lattices belong to the same universality class as the Sierpinski gasket embedded in two and three dimensions respectively.

3. The truncated 5-simplex lattice

The basic geometrical unit of the construction of a truncated 5-simplex lattice is a pentagon with 5-corner vertices and bonds between every pair of vertices. Each vertex connected through a direct bond is termed a nearest neighbour. The pentagons of first

and $(r + 1)$ th orders are shown in figure 1. One finds seven restricted partition functions (figure 2) which should be sufficient for representing the generating functions $P(x)$, $C(x)$ and $R(x)$ at an arbitrary stage of the coarse graining process. The starting values (pertinent to a unit pentagon) of these functions are

$$\begin{aligned} A_0 &= x & C_0 &= \sqrt{x} \\ C_0 &= D_0 = E_0 = F_0 = 0 \end{aligned} \tag{3.1}$$

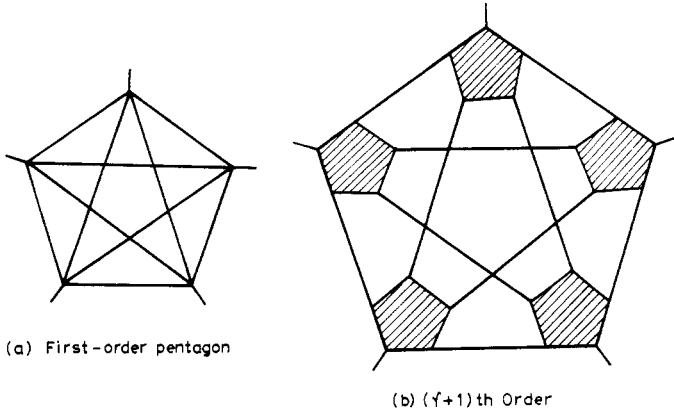


Figure 1. Graphical representation of a truncated 5-simplex lattice. (a) Graph of a first-order pentagon. (b) Schematic representation of the graph of the $(r + 1)$ th order. The shaded pentagons denote the graph at r th-order pentagons of which only the corner vertices are shown.

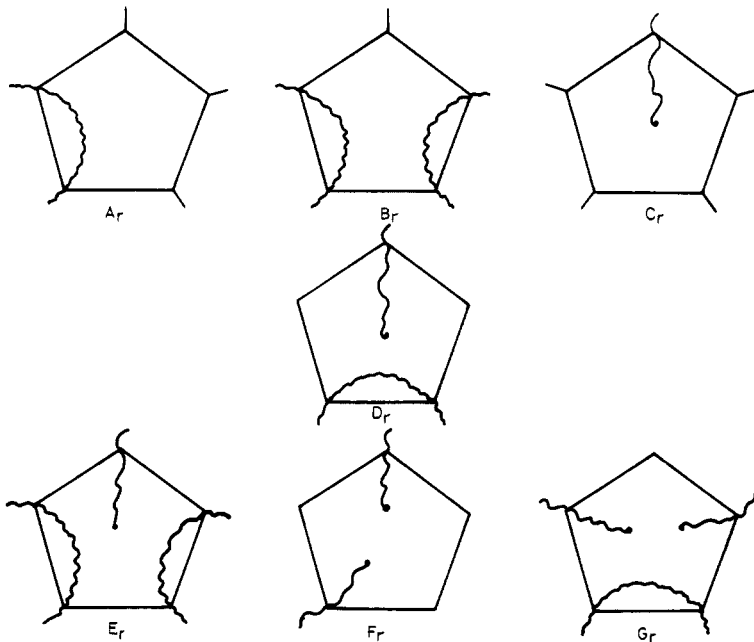


Figure 2. A diagrammatic representation of the seven restricted generating functions for an r th-order pentagon. The interior structure of the pentagon is not shown. Only the corner vertices and the end points of the self-avoiding walks are shown.

where their r th-order values are the sum (weights) of all possible walks within the r th-order pentagon consistent with the constraint shown in figure 2. The recursion relations for these functions are

$$A_{r+1} = f_a(A_r, B_r) \tag{3.2a}$$

$$B_{r+1} = f_b(A_r, B_r) \tag{3.2b}$$

$$C_{r+1} = f_c(A_r, B_r, \dots, E_r) \tag{3.2c}$$

$$D_{r+1} = f_d(A_r, B_r, \dots, E_r) \tag{3.2d}$$

$$E_{r+1} = f_e(A_r, B_r, \dots, E_r) \tag{3.2e}$$

$$F_{r+1} = f_f(A_r, B_r, \dots, G_r) \tag{3.2f}$$

$$G_{r+1} = f_g(A_r, B_r, \dots, G_r). \tag{3.2g}$$

The forms of functions f_a, \dots, f_e were generated on a computer, and are given in appendix A. The forms of f_f and f_g are not needed in the evaluation of the critical exponents and, therefore, are not given here. The A and B coordinates of a non-trivial fixed point (A^*, B^*, \dots, G^*) of the RG transformations of (3.2) are found by solving (A1), (A2) independently of the similar RG equations (A3) for the other five coordinates. The exponent ν is determined by the coordinates A and B only. The connectivity constant μ is also determined by the A^* and B^* coordinates of the fixed point. We therefore first consider the solution of (A1), (A2).

The generating function $P(x)$ is written as

$$P(x) = \sum_{r=1}^{\infty} \frac{1}{5^r} f_p(A_{r-1}, B_{r-1}) \tag{3.3}$$

where f_p is a polynomial in A_{r-1}, B_{r-1} and their combinations. An explicit form of the polynomial is not needed. For a lattice having a finite ramification number, the coefficients of the polynomial are always finite. If A_{r-1} and B_{r-1} are less than the respective coordinates of the fixed point (A^*, B^*) , successive terms in (3.3) decrease because A_r and B_r tend to zero with large r and the series converges. But if the value of A_{r-1} and B_{r-1} are larger than A^* and B^* , respectively, the successive terms in (3.3) increase and $P(x)$ diverges. The value of $x(=x^*)$ corresponding to the fixed point (A^*, B^*) determines the connectivity constant, i.e. $\mu = 1/x^*$.

Starting with $(A, B) = (x, 0)$ we find

$$x^* = 0.336\ 017 \tag{3.4}$$

and

$$A^* = 0.3265 \tag{3.5}$$

$$B^* = 0.02791. \tag{3.6}$$

There are other fixed points of the recursion (3.3). The fixed points $A^* = B^* = 0$ and $A^* = B^* = \infty$ are the trivial. From (3.4) we get $\mu = 2.976\ 03$. Linearisation of the recursion relations (A1), (A2) around the fixed point leads to one eigenvalue

$$\lambda_1 = 3.131\ 99 \tag{3.7}$$

greater than unity. The eigenvectors corresponding to λ_1 determine the exponent ν . Therefore

$$\nu = \frac{\ln b}{\ln \lambda_1} = 0.604\ 905 \tag{3.8}$$

and

$$\alpha = 2 - \frac{\ln n}{\ln \lambda_1} = 0.595\ 47. \tag{3.9}$$

Here $b = 2$. As expected the value of ν is lower than that of the 4-simplex lattice (=0.6740).

In order to calculate the exponent γ following Dhar [15], we choose a small positive number ε and choose a value of x sufficiently close ($\sim \delta$) to $x^*(=\mu^{-1})$ so that

$$1 \gg \varepsilon \gg \delta \tag{3.10}$$

and

$$r_0 = \ln(\varepsilon/\delta)/\ln \lambda_1. \tag{3.11}$$

Thus for $r < r_0$ we have $A_r = A^*$, $B_r = B^*$ and the recursion equations for C_{r+1} , D_{r+1} and E_{r+1} become, from (A3), (A4) and (A5),

$$C_{r+1} = c_1(A^*, B^*)C_r + c_2(A^*, B^*)D_r + c_3(A^*, B^*)E_r \tag{3.12a}$$

$$D_{r+1} = d_1(A^*, B^*)C_r + d_2(A^*, B^*)D_r + d_3(A^*, B^*)E_r \tag{3.12b}$$

$$E_{r+1} = e_1(A^*, B^*)C_r + e_2(A^*, B^*)D_r + e_3(A^*, B^*)E_r. \tag{3.12c}$$

Since the forms of the polynomials c_i , d_i and e_i appearing in the above equations can easily be found from (A3)-(A5), they are not written here explicitly. Equations (3.12a-c) are linear recursion equations and show that, for $r < r_0$, C_r , D_r and E_r increase as λ_+^r , where λ_+ is the largest eigenvalue of the matrix and characterises the linear transformation of (3.12). λ_+ has been found equal to 5.243 98.

For $r > r_0$ the coefficients A_r , B_r , D_r and E_r rapidly approach zero and C_r tends to its asymptotic value which is proportional to $\lambda_+^{r_0}$. Dhar [15] has shown that

$$C(x) \sim K \left(\frac{\lambda_+^2}{n} \right)^{r_0} \tag{3.13}$$

which leads to

$$\gamma = \ln(\lambda_+^2/n)/\ln \lambda_1. \tag{3.14}$$

For the 5-simplex lattice, we have $\gamma = 1.4875$.

4. 6-simplex lattice

The basic geometrical unit of the construction of a truncated 6-simplex lattice is a hexagon with six-corner vertices and bonds between every pair of vertices. The hexagons of first and $(r + 1)$ th orders are shown in figure 3. Any two vertices connected through direct bond are nearest neighbours. The nine restricted generating functions corresponding to this lattice are shown in figure 4. The starting values of these functions are

$$\begin{aligned} A_0 &= x & D_0 &= \sqrt{x} \\ B_0 &= C_0 = E_0 = \dots = I_0 & &= 0 \end{aligned} \tag{4.1}$$

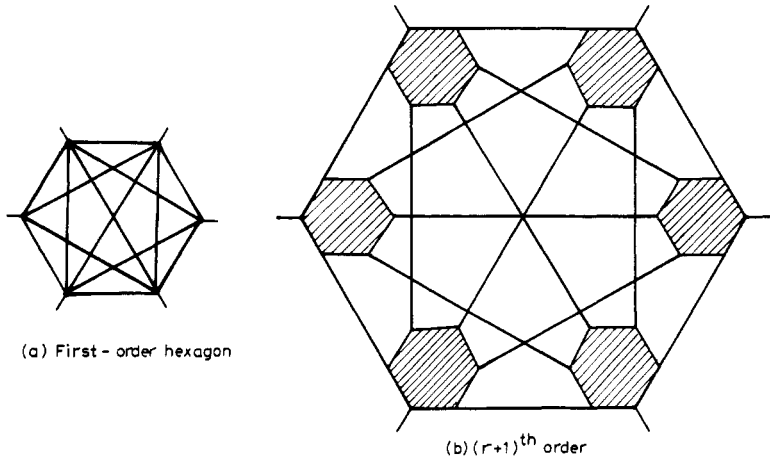


Figure 3. Graphical representation of a truncated 6-simplex lattice. (a) Graph of a first-order hexagon. (b) Schematic representation of the graph of the $(r+1)$ th order. The shaded hexagons denote the graph of r th-order hexagons of which only the corner vertices are shown.

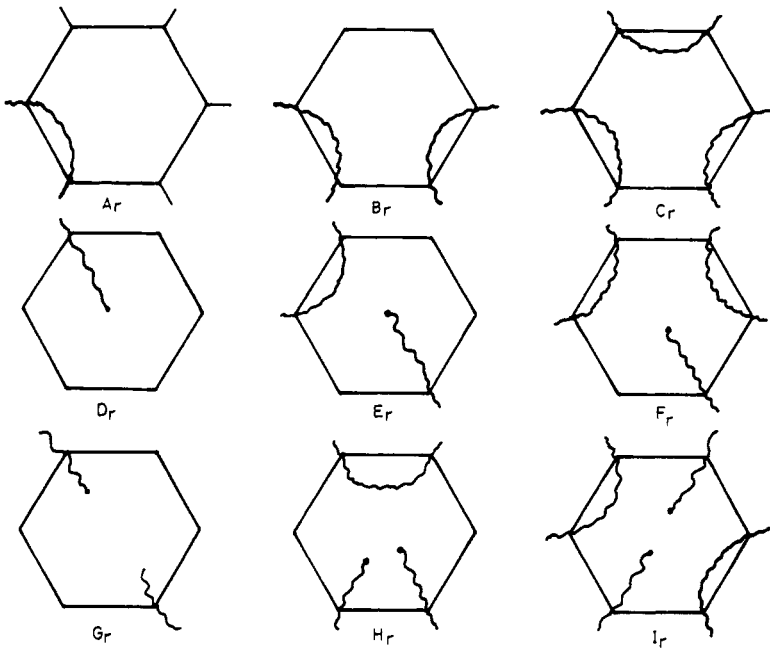


Figure 4. A diagrammatic representation of the nine restricted generating function of an r th-order hexagon. The remainder is as in figure 2.

where their r th-order values are sums of all possible walks within the r th-order hexagon consistent with the constraint shown in figure 4. The recursion relations for these functions are given in appendix B. As argued in the case of the 5-simplex lattice, the recursion relations for A_{r+1} , B_{r+1} and C_{r+1} involve only A_r , B_r , C_r and their combinations and are independent of other functions.

The (A^*, B^*, C^*) coordinates of a non-trivial fixed point $(A^*, B^*, C^*, \dots, I^*)$ of the RG transformation are, therefore, found by solving the above recursion relations independently of the similar RG equations for other six coordinates. Starting with $(x, 0, 0)$ we get the fixed point

$$\begin{aligned} A^* &= 0.262\ 352 \\ B^* &= 0.017\ 588 \\ C^* &= 0.000\ 7011 \end{aligned} \tag{4.2}$$

and the corresponding fugacity $x^* = 0.271\ 66$. The connectivity constant $\mu = 3.681\ 07$.

The linearisation of the recursion relations (B1), (B3) about this fixed point yields only one eigenvalue, $\lambda_1 = 3.4965$, for which the value is higher than unity. Thus

$$\nu = \frac{\ln 2}{\ln 3.4965} = 0.5537 \tag{4.3}$$

and

$$\alpha = 2 - \frac{\ln 6}{\ln 3.4965} = 0.5686. \tag{4.4}$$

Following the method described briefly in the preceding section and the relations for D_{r+1} , E_{r+1} and F_{r+1} given in appendix B (B4), (B6), we find the matrix which characterises the linear transformation of the recursion relation of these functions and for which the largest eigenvalue, $\lambda_+ = 6.267\ 09$. Using relation (3.14) we find that for a 6-simplex lattice

$$\gamma = 1.500\ 094.$$

5. An approximate theory for the critical exponents of SAWs on n -simplex lattices

Here we describe an approximate theory for calculating the critical exponents of SAWs on n -simplex lattices when $n \rightarrow \infty$. For this we first note that for large n , the coordinates of the fixed point corresponding to a swollen state behave as

$$A^* \sim \frac{1}{n} \quad B^* \sim \frac{1}{n^2} \quad C^* \sim \frac{1}{n^3}. \tag{5.1}$$

Thus B^*, C^*, \dots approach zero faster than A^* as n is increased. Therefore, for calculating ν for large n we consider the generating function for the configuration A only and neglect other configurations, i.e. put $B_r = C_r = \dots = 0$ for all values of r . This leads to following simple recursion relation for A :

$$D_{r+1} = A^2 + (n-2)A^3 + (n-2)(n-3)A^4 + \dots + (n-2)! A^n. \tag{5.2}$$

Similarly, for calculating γ we consider the functions A and D (of figure 3) and put the others equal to zero. The recursion relation for D is found to be

$$D_{r+1} = D_r [1 + (n-1)A^* + (n-1)(n-2)A^{*2} + \dots + (n-1)! A^{*n-1}] \tag{5.3}$$

where A^* is the fixed point of (5.2). Using these relations we calculate μ, ν, α and γ for a large number of lattices. The results which are designated with subscript A1, are given in table 1. Though as mentioned above, this method is expected to give accurate

Table 1. The critical exponents (ν, γ, α) and the connectivity constants (μ) together with the eigenvalues λ_{A1} for SAWs on family truncated n -simplex lattices as obtained from the approximate theory of section 5 are given. The fractal dimension \tilde{d} and the spectral dimension \bar{d} are also given.

n	\tilde{d}	\bar{d}	$x^* = 1/\mu$	λ_{A1}	ν_{A1}	γ_{A1}	α_{A1}
3	1.3652	1.5849	0.6180	2.3819	0.798	1.265	0.7342
4	1.5474	2.00	0.4406	2.7304	0.690	1.380	0.6198
5	1.6541	2.3219	0.3395	3.0548	0.620	1.441	0.5588
6	1.7233	2.5849	0.2747	3.3604	0.571	1.478	0.5217
7	1.7712	2.8073	0.2299	3.6509	0.535	1.502	0.4973
8	1.8061	3.00	0.1971	3.9288	0.506	1.519	0.4805
9	1.8326	3.1699	0.1722	4.1958	0.483	1.532	0.4678
10	1.8532	3.3219	0.1527	4.4550	0.463	1.541	0.4588
15	1.9116	3.9068	0.0964	5.6339	0.400	1.566	0.4336
16	1.9185	4.00	0.0897	5.8529	0.392	1.569	0.4308
20	1.9383	4.3219	0.0698	6.6852	0.364	1.576	0.4232
30	1.9627	4.9068	0.0445	8.5425	0.323	1.585	0.4144
40	1.9738	5.3219	0.0324	10.1836	0.298	1.589	0.4105
50	1.9801	5.6438	0.0254	11.6772	0.282	1.591	0.4082
60	1.9841	5.9068	0.0208	13.0607	0.269	1.593	0.4066
70	1.9868	6.1292	0.0176	14.3591	0.260	1.594	0.4054
80	1.9887	6.3219	0.0152	15.5872	0.252	1.595	0.4045
90	1.9902	6.4918	0.0134	16.7567	0.245	1.596	0.4036
100	1.9914	6.6438	0.0120	17.8761	0.240	1.597	0.4029
200	1.9962	7.6438	0.0057	27.3043	0.209	1.602	0.3979
300	1.9976	8.2288	0.0037	34.9135	0.195	1.605	0.3946
400	1.9983	8.6438	0.0027	41.5193	0.186	1.607	0.3920
500	1.9987	8.9657	0.0022	47.4702	0.179	1.609	0.3900
1000	1.9994	9.9657	0.0010	71.6458	0.162	1.617	0.3829

results for large n , we give results for small n also so that comparison can be made with exact values. The following observations are in order. (i) The values of ν are close to those found exactly. For $n=3$, (5.3) gives an exact recursion relation. The error, however, increases initially as n increases but, as argued above, should decrease with higher values of n . (ii) The value of $\nu \rightarrow 0$ as $n \rightarrow \infty$. (iii) The values of exponent γ known exactly for $n=3, \dots, 6$ are close to the value found here. The error in γ decreases rapidly as n increases. (iv) As $n \rightarrow \infty$, $\gamma \sim 1.618$, a constant.

The series of (5.2) can be approximated (for large n) to give

$$A_{r+1} \approx (n-2)! A_r^n e^{1/A}. \tag{5.4}$$

The fixed point of this relation is found to be

$$A^* = \frac{1+s}{n+1} \tag{5.5}$$

where

$$s \approx \frac{1}{n} + \sqrt{\frac{\ln(n+2)}{n}}.$$

However, we note that the relation

$$A^* = \frac{1}{n+1} \left(1 - \frac{1}{(n+1)^{1-p}} \right)^{-1} \tag{5.6}$$

where $p = 0.6180 \approx (\sqrt{5} - 1)/2$ (this value represents the fixed point of the 3-simplex lattice) gives a better representation of the fixed points of (5.2).

The linearisation of (5.2) about its fixed points give eigenvalues

$$\lambda_1 = (n + 1) - \frac{1}{A^*} \tag{5.7}$$

which determine the exponent ν . From (5.6) and (5.7) we find

$$\lambda_1 = (n + 1)^p. \tag{5.8}$$

This simple looking relation is found to scale the eigenvalues very well for all values of n . From (5.8) and the definition of the exponent ν we find the following appropriate relation:

$$\frac{\tilde{d}}{\nu} = p\{2\tilde{d} - \tilde{d}/[2 \ln 2(2^{\tilde{d}-1} + 1)]\}. \tag{5.9}$$

For large n (5.9) reduces to a simple form

$$\nu \sim \frac{\tilde{d}}{2p\tilde{d}}. \tag{5.10}$$

The relation (5.9) and (5.10) (for large n only) give a very good representation of ν found from the numerical solution of (5.2).

It may be noted that the eigenvalue found from (5.3) at the fixed point A^* is given as

$$\lambda_+ = n. \tag{5.11}$$

This leads to a very simple scaling relation between ν , α and γ :

$$\gamma = \nu\tilde{d}$$

and

$$\alpha + \gamma = 2. \tag{5.12}$$

These relations are very well satisfied by the values in table 1. Further, we note from (5.10) and (5.12) that for large n ($\tilde{d} \approx 2$)

$$\gamma \sim \frac{1}{p} = 1.618. \tag{5.13}$$

This is an interesting result. It indicates that the parameter p which appeared in (5.6) is the inverse of γ in the limit $1/n \rightarrow 0$.

The approximate theory given here can be improved systematically by including terms corresponding to other configurations. For example, we can include terms linear in B in (5.2) and write a recursion relation for B which has terms involving only A . This leads to following relations:

$$A_{r+1} = A(A + (n - 2)A^2 + \dots + (n - 2)A^{n-1} + B(2(n - 2)(n - 3)A^3 + 5(n - 2)(n - 3)(n - 4)A^4 + \dots)) \tag{5.14a}$$

and

$$B_{r+1} = A^4 + 2(n - 4)A^5 + 3(n - 4)(n - 5)A^6 + \dots \tag{5.14b}$$

These recursion relations give values of ν which are somewhat lower than those given by (5.2) for small values of n , but become almost identical for $n \geq 150$. The difference in the values of ν found from these two equations is largest around $n \sim 30$.

6. Discussion

In figure 5 we compare the values of ν obtained from (5.2) (or (5.9)) designated ν_{A1} and from (5.4a, b) designated ν_{A2} with the exact results. We also give in this figure values of ν found from phenomenological relations of (6.1), (6.2) and (6.4) given below. We find that (5.4a, b) give values of ν which are very close to exact values even for small values of n . Thus the method described in the above section provides a very simple and reasonably accurate recipe for calculating critical exponents for SAWs on fractals.

Motivated by the Flory formula (1.1) several attempts have been made in recent years to find closed relations for critical exponents [16, 25, 26] of SAWs on fractals. The following phenomenological relations were proposed for ν :

$$\nu_K = \frac{3}{3 + 2\bar{d}} \tag{6.1}$$

$$\nu_{RTV} = \frac{3\bar{d}}{\bar{d}(2 + \bar{d})} \tag{6.2}$$

and

$$\nu_A = \frac{4 + \bar{d}}{4\bar{d}}. \tag{6.3}$$

Relation (6.1) was proposed by Kramer [25] and Sahimi [26] and appears to be a straightforward generalisation of the self-consistent Flory formula (1.1) to fractals.

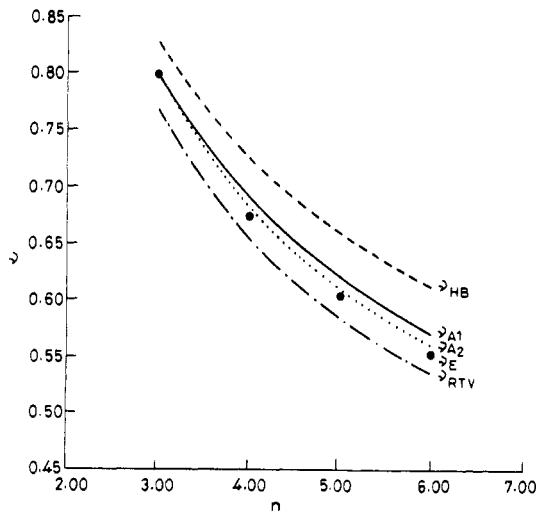


Figure 5. A comparison of the results obtained from exact calculations ν_E (full circles) with ν_{A1} (full curve), ν_{A2} (dotted curve), and ν_{RTV} (chain curve) and ν_{HB} (short broken curve).

Equation (6.2) was given independently by Rammal *et al* [16] and Sahimi [26]. The last relation (6.3) is due to Alexandrowicz [27] and is valid for $4/3 \leq \bar{d} \leq 4$. Eležović *et al* [17] have found that (6.3) highly overestimates ν . We find that while (6.2) gives values for ν they are somewhat underestimated yet are consistent for all values of n . Equation (6.1) underestimates the values of ν for small values of n and highly overestimates at large values of n .

A more sophisticated relation than given above ((6.1)-(6.3)) has recently been proposed by Halvin and Ben Avraham [28] and justified by Bouchaud and Georges [29] and Aharony and Harris [30]. Their relation involves the spreading dimension, d_s , [31] in addition to \bar{d} and \tilde{d} , i.e.

$$\nu_{HB} \bar{d} = \frac{4d_s - \tilde{d}}{2 + 2d_s - \tilde{d}} \tag{6.4}$$

Taking $d_s = \bar{d}$ for the n -simplex lattices we find that (6.4) somewhat overestimates the value of ν [29] though is better than (6.1) and (6.3).

To demonstrate the correctness of the relations (6.1)-(6.4) we plot in figure 6 $\alpha (=2 - \nu \bar{d})$ as a function of $\ln n$. The exact values of α found for $n = 3, 4, 5$ and 6 are also shown in the figure. We find that our approximate theory gives reasonably good values of α though somewhat underestimated. This is a consequence of ν being somewhat overestimated. As expected, the broken curve, which corresponds to the values of ν designated ν_{A2} in figure 5 and is found from (5.14a, b), gives better agreement with the exact results for $n = 5$ and 6 as obtained in this paper and also for $n = 3$ and 4 as obtained by Dhar [15]. We also find that the Rammal *et al* [16] relation, although it overestimates α for all values of \bar{d} , gives a qualitatively better result than that of the relation corresponding to (6.1) which makes α negative rapidly as \bar{d} is increased. Equation (6.4) gives values for α which are always underestimated. The error increases with n .

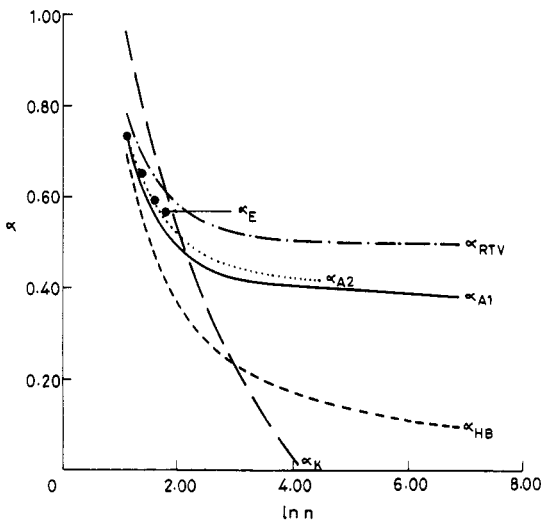


Figure 6. A comparison of the results of the critical exponent α obtained from the exact calculations with those obtained from the approximate theory and from phenomenological relations. Labelling is the same as in figure 5.

Attempts have also been made to find a closed expression for the exponent γ . Alexandrowicz [27] found the relation

$$\gamma_A = \frac{8}{4 + \bar{d}} \tag{6.5}$$

from random walks through a progressive exclusion of walks with loops and is connected with (6.3) by the relation $\gamma\nu\bar{d} = 2$. The other relation

$$\gamma_{PP} = \frac{6}{2 + \bar{d}} \tag{6.6}$$

has been suggested by Pietronero and Pelity [32].

In figure 7 we plot the values of γ obtained from the above relations and from the approximate theory of section 5 (or from (5.9) and (5.12)). We also plot in the figure the known exact values. The values of γ found from the approximate theory are in good agreement with the exact values and also exhibit variation with \bar{d} which is similar to those found by Eležovič *et al* [17]. However, the relations of (6.5) and (6.6) give values of γ which decrease with \bar{d} (or $\ln n$) which is in contradiction with known behaviour of γ with \bar{d} .

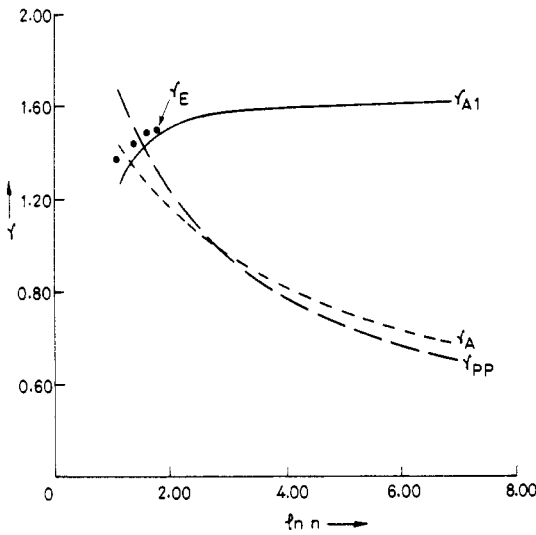


Figure 7. The critical exponent γ as a function of $\ln n$. The full curve represents the values (γ_{A1}) obtained from the approximate theory of section 5 and the short broken curve, γ_A (6.4) and the long broken curve, γ_{PP} (6.5). Exact values (γ_E) are shown by full circles. Exact values are very close to the full curve.

In conclusion we wish to emphasise that none of the phenomenological relations given in the literature for the critical exponents of SAWs on fractals represents them correctly. The relation (6.2) for ν and the corresponding relation for α appears to be better than others. The suggestions that if \bar{d} and \bar{d} are equal, the critical exponents on fractals are an analytical continuation of their values on a regular integer-dimensional lattice [33, 34] has recently been found not to be correct by Dhar [22].

A very simple approximate theory given in section 5 appears to give reasonable values of the critical exponents even for small values of n where the error is expected to be large. The exponents are very well approximated by (5.9) and (5.12). These equations appear to be best among the suggested relations for the family of fractals studied in this paper. The predictive power of the theory can further be improved by systematically incorporating more terms in (5.2) and (5.3).

Acknowledgments

We are grateful to Deepak Dhar for many stimulating discussions and valuable suggestions during the progress of this work and to the referee of the paper for bringing to our attention the work of [27-29]. We also acknowledge financial support by the Department of Science and Technology (India). One of us (SK) thanks UGC, New Delhi for providing a research fellowship.

Appendix A

Here we report the recursion relations found for the 5-simplex lattice

$$A_{r+1} = A^2 + 3A^3 + 6A^4 + 6A^5 + 132AB^4 + 132B^5 + 18A^2B^2 + 96A^2B^3 + 12A^3B + 78A^3B^2 + 30A^4B \quad (\text{A1})$$

$$B_{r+1} = A^4 + 2A^5 + 13A^4B + 32A^3B^2 + 4A^3B + 88A^2B^3 + 220AB^4 + 186B^5 + 22B^4 \quad (\text{A2})$$

$$C_{r+1} = C(1 + 4A + 12A^2 + 24A^3 + 24A^4 + 24A^2B + 72A^2B^2 + 72A^3B) + D(12A^2 + 48A^3 + 72A^4 + 72A^2B + 456A^2B^2 + 288A^3B + 528AB^3 + 528AB^4) + E(24A^4 + 96A^3B + 528B^4) \quad (\text{A3})$$

$$D_{r+1} = C(A^2 + 4A^3 + 6A^4 + 6A^2B + 24A^3B + 38A^2B^2 + 44AB^3 + 44B^4) + D(3A^2 + 14A^3 + 26A^4 + 32A^2B + 140A^3B + 44AB^2 + 374A^2B^2 + 636AB^3 + 44B^3 + 548B^4) + E(12A^4 + 72A^3B + 208A^2B^2 + 592AB^3 + 856B^4) \quad (\text{A4})$$

$$E_{r+1} = C(A^4 + 4A^3B + 22B^4) + D(6A^4 + 36A^3B + 104A^2B^2 + 296AB^3 + 428B^4) + E(5A^4 + 472AB^3 + 1042B^4). \quad (\text{A5})$$

The subscript r has been dropped from the right-hand side of the above equations.

Appendix B

Here we present the recursion relations found for the 6-simplex lattice.

$$\begin{aligned}
 A_{r+1} = & A^2(1+36B^2+384B^3+5544B^4+4992B^3C)+4A^3(1+78B^2+648B^3+432B^2C) \\
 & +12A^4(1+10B+80B^2+10C^2+40BC)+24A^5(1+2C+9B) \\
 & +24AB^3(1+22B+362B^2+636BC+472C^2) \\
 & +48B^4(11B+137B^2+521C^2+428BC)
 \end{aligned} \tag{B1}$$

$$\begin{aligned}
 B_{r+1} = & A^4(1+26B+144BC+324B^2+6A^2)+4A^5(1+4C+16B) \\
 & +4A^2B^2(44B+905B^2+1272BC+708C^2) \\
 & +4A^3B(1+16B+308B^2+208BC) \\
 & +8AB^3(55B+822B^2+2140BC+2084C^2) \\
 & +B^2(22B^2+372B^3+5440B^4+23\,520B^3C+48\,160B^2C^2 \\
 & +76\,800BC^3+94\,336C^4)
 \end{aligned} \tag{B2}$$

$$\begin{aligned}
 C_{r+1} = & A^6+6A^5(C+2B)+4A^3B^2(9A^2+52AB)+18A^2B^3(159B+236C) \\
 & +AB^4(2568B+6252C)+2940B^6+14\,448B^5C+43\,200B^4C^2 \\
 & +94\,336B^3C^3+541\,568C^6
 \end{aligned} \tag{B3}$$

$$\begin{aligned}
 D_{r+1} = & D[1+20A^2(1+3B+30B^2+114B^3)+60A^3(1+6B+8C+30B^2) \\
 & +120A^4(1+C+6B)+5A(1+24A^4+528B^4)+2640B^4(A+B+C)] \\
 & +E[20A^2(1+9B+114B^2+624CB^2+1350B^3) \\
 & +120A^3(1+12B+36BC+102B^2) \\
 & +120A^4(3+30B+6C+4A)+240B^3(11+148C) \\
 & +240B^4(11+181A+148B+214C)]+F[120A^3(3A^2+4AB+68B^2) \\
 & +120A^4(1+5C+23B)+960AB^3(24A+59C) \\
 & +240B^4(11+269A+307B)+521C]
 \end{aligned} \tag{B4}$$

$$\begin{aligned}
 E_{r+1} = & D[A^2(1+9B+114B^2+1350B^3+624CB^2)+6A^3(1+12B+102B^2+26BC) \\
 & +6A^4(3+4A+30B+6C)+4AB^3(33+444C) \\
 & +4B^4(33+543A+444B+642C)] \\
 & +E[3A^2(1+16B+374B^2+5120B^3+1416BC^2+4872CB^2) \\
 & +3A^3(7+140B+1786B^2+976BC+152C^2) \\
 & +6A^4(13+21A+197B+50C)+11AB^2(6+318B+4168C^2 \\
 & +7592BC)+6B^2(11B+9600C^3+1380BC^2) \\
 & +12B^4(137+2314A+1974B+5562C)] \\
 & +F[12A^4(3+10A+103B+28C) \\
 & +48AB^3(37+1197B+2169C) \\
 & +1200C^2)+24A^2B^2(26+994B+865C) \\
 & +24A^3B(9+265B+120C)+24B^4(107+2555B+8814C) \\
 & +384B^2C^2(887B+736C)]
 \end{aligned} \tag{B5}$$

$$\begin{aligned}
F_{r+1} = & D[A^4(1+3A+23B+5C)+4AB(A^2+17A^2B+48AB^2+118CB^2) \\
& +2B^4(11+269A+307B+521C)] \\
& +E[2A^4(3+10A+103B+28C)+8AB^3(37+1197B+2169C \\
& +1200C^2)+8A^2B^2(13+497B+455C) \\
& +4A^3B(9+265B+40C)+64B^2C^2(887B+737C) \\
& +4B^4(107+2555B+8814C)] \\
& +F[A^4(5+21A+168B)+2AB^3(236+12793B+29232C) \\
& +4A^2B(1927B^2+11792C^2+1692BC) \\
& +4B^2(297A^3+111808C^3+80816BC^2) \\
& +2B^4(521+74000C+16580B)+541568C^4(B+C)]. \quad (B6)
\end{aligned}$$

Since the recurrence relations of function G , H and I are not needed for the evaluation of the critical exponents of the SAW, they are not given here.

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